

Exactness, K-property and infinite mixing

MARCO LENCI ^{*†}

December 2012

Abstract

We explore the consequences of exactness or K-mixing on the notions of mixing (a.k.a. *infinite-volume mixing*) recently devised by the author for infinite-measure-preserving dynamical systems.

Mathematics Subject Classification (2010): 37A40, 37A25.

1 Introduction

Currently, in infinite ergodic theory, there is a renewed interest in the issues related to mixing for infinite-measure-preserving (or just nonsingular) dynamical systems, in short *infinite mixing* (see [Z, DS, L1, I3, DR, MT, LP, A2, Ko, T1], and some applications in [I1, I2, AMPS, L2, T2]).

The present author recently introduced some new notions of infinite mixing, based on the concept of *global observable* and *infinite-volume average* [L1]. In essence, a global observable for an infinite, σ -finite, measure space (X, \mathcal{A}, μ) is function in $L^\infty(X, \mathcal{A}, \mu)$ that “looks qualitatively the same” all over X , and is thus far from being integrable. We refer to the latter type of function as *local observable*.

Postponing the mathematical details to Section 2, the motivation for this new class of objects is basically twofold. First, the past attempts to a general definition of infinite mixing involved mainly local observables (equivalently, finite-measure sets), and the problems with such definitions seemed to depend on that. Second, seeking inspiration in statistical mechanics (which is the discipline of mathematical physics that has successfully dealt with the question of predicting measurements in very large, formally infinite, systems), one realizes that many quantities of interest are *extensive observables*, that is, objects that behave qualitatively in the same way in

^{*}Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy. E-mail: marco.lenci@unibo.it

[†]INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy.

different regions of the phase space. (More detailed discussions about these points are found in in [L1, L2].)

Extensive observables are “measured” by taking averages over large portions of the phase space. We import that concept too, by defining the infinite-volume average of a global observable $F : X \rightarrow \mathbb{R}$ as

$$\bar{\mu}(F) := \lim_{V \nearrow X} \frac{1}{\mu(V)} \int_V F d\mu. \quad (1.1)$$

Here V is taken from a family of ever larger but finite-measure sets that somehow covers, or *exhausts* the whole of X . The precise meaning of the limit above will be given in Section 2.

Now, let us consider a measure-preserving dynamical system on (X, \mathcal{A}, μ) . For the sake of simplicity, let us restrict to the discrete-time case: this means that we have a measurable map $T : X \rightarrow X$ that preserves μ . Choosing two suitable classes of global and local observables, respectively denoted \mathcal{G} and \mathcal{L} , we give five definitions of infinite mixing. These fall in two categories, exemplified as follows.

Using the customary (abuse of) notation $\mu(g) := \int_X g d\mu$, we say that the system exhibits:

- *global-local mixing* if, $\forall F \in \mathcal{G}, \forall g \in \mathcal{L}, \lim_{n \rightarrow \infty} \mu((F \circ T^n)g) = \bar{\mu}(F)\mu(g)$;
- *global-global mixing* if, $\forall F, G \in \mathcal{G}, \lim_{n \rightarrow \infty} \bar{\mu}((F \circ T^n)G) = \bar{\mu}(F) \bar{\mu}(G)$.

Disregarding for the moment the mathematical issues connected to the above notions, we focus on the interpretation of global-local mixing. Restricting, without loss of generality, to local observables $g \geq 0$ with $\mu(g) = 1$, and defining $d\mu_g := g d\mu$, the above limit reads:

$$\lim_{n \rightarrow \infty} T_*^n \mu_g(F) = \bar{\mu}(F), \quad (1.2)$$

where the measure $T_*^n \mu_g$ is the push-forward of μ_g via the dynamics T^n (in other words, $T_*^n \mu_g := \mu_g \circ T^{-n} = \mu_{P^n g}$, where P is the Perron-Frobenius operator relative to μ , cf. (3.2)-(3.3)). If (1.2) occurs for all $g \in \mathcal{L}$ and $F \in \mathcal{G}$, the above is a sort of “convergence to equilibrium” for all *initial states* given by μ -absolutely continuous probability measures. In this sense the functional $\bar{\mu}$ (not a measure!) plays the role of the *equilibrium state*.

Exactness and K-mixing (a.k.a. the K-property) are notions that exist and have the same definition both in finite and infinite ergodic theory. In finite ergodic theory they are known to be very strong properties, as they imply mixing of all orders, cf. definition (3.1). The purpose of this note is to explore their implications in terms of the notions of infinite mixing introduced in [L1].

As we will see below (Theorem 3.5(a)), the most notable of such implications is a weak form of global-local mixing, whereby any pair of measure μ_g, μ_h , as introduced earlier, are *asymptotically coalescing*, in the weak sense that

$$\lim_{n \rightarrow \infty} (T_*^n \mu_g(F) - T_*^n \mu_h(F)) = 0, \quad (1.3)$$

for all $F \in \mathcal{G}$.

In the next section we review the five definitions of global-local and global-global mixing, together with the already known (though with a different name) definition of local-local mixing. In Section 3 we prepare, state and prove Theorem 3.5, which lists some consequences of exactness and the K-property. Finally, in Section 4, we introduce the space of the *equilibrium observables*, which is a purely ergodic-theoretical construct in which some information about global-local mixing can be recast.

2 Definitions of infinite mixing

Let (X, \mathcal{A}, μ, T) be a measure-preserving dynamical system, where (X, \mathcal{A}) is a measure space, μ an infinite, σ -finite, measure on it, and T a μ -endomorphism, that is, a measurable surjective map that preserves μ (i.e., $\mu(T^{-1}A) = \mu(A)$, $\forall A \in \mathcal{A}$).

Denoting by $\mathcal{A}_f := \{A \in \mathcal{A} \mid \mu(A) < \infty\}$ the class of finite-measure sets, we assume that the following additional structure is given for the dynamical system:

- A class of sets $\mathcal{V} \subset \mathcal{A}_f$, called the **exhaustive family**. The elements of \mathcal{V} will be generally indicated with the letter V .
- A subspace $\mathcal{G} \subset L^\infty(X, \mathcal{A}, \mu; \mathbb{R})$, whose elements are called the **global observables**. These functions are indicated with uppercase Roman letters (F, G , etc.).
- A subspace $\mathcal{L} \subset L^1(X, \mathcal{A}, \mu; \mathbb{R})$ whose elements are called the **local observables**. These functions will be indicated with lowercase Roman letters (f, g , etc.).

A discussion on the role and the choice of $\mathcal{V}, \mathcal{G}, \mathcal{L}$ is given in [L1], together with the proofs of most assertions made in this section.

We assume that \mathcal{V} contains at least one sequence $(V_j)_{j \in \mathbb{N}}$, ordered by inclusion, such that $\bigcup_j V_j = X$. (In actuality, this requirement is never used in the proofs, but, since the elements of \mathcal{V} are regarded as large and “representative” regions of the phase space X , we keep it to give “physical” meaning to the concept of infinite-volume average, see below.) We also assume that $1 \in \mathcal{G}$ (with the obvious notation $1(x) := 1$, $\forall x \in X$).

Definition 2.1 *Let \mathcal{V} be the aforementioned exhaustive family. For $\phi : \mathcal{V} \rightarrow \mathbb{R}$, we write*

$$\lim_{V \nearrow X} \phi(V) = \ell$$

when

$$\lim_{M \rightarrow \infty} \sup_{\substack{V \in \mathcal{V} \\ \mu(V) \geq M}} |\phi(V) - \ell| = 0.$$

We call this the ‘ μ -uniform infinite-volume limit w.r.t. the family \mathcal{V} ’, or, for short, the **infinite-volume limit**.

We assume that, $\forall n \in \mathbb{N}$,

$$\mu(T^{-n}V \triangle V) = o(\mu(V)), \quad \text{as } V \nearrow X. \quad (2.1)$$

This is reasonable because, if a large $V \in \mathcal{V}$ is to be considered a finite-measure substitute for X , it makes sense to require that a finite-time application of the dynamics does not change it much. Finally, the most crucial assumption is that,

$$\forall F \in \mathcal{G}, \quad \exists \bar{\mu}(F) := \lim_{V \nearrow X} \frac{1}{\mu(V)} \int_V F d\mu. \quad (2.2)$$

$\bar{\mu}(F)$ is called the **infinite-volume average** of F w.r.t. μ . It is easy to check that $\bar{\mu}$ is T -invariant, i.e., for all $F \in \mathcal{G}$ and $n \in \mathbb{N}$, $\bar{\mu}(F \circ T^n)$ exists and equals $\bar{\mu}(F)$ [L1].

With this machinery, we can give a number of definitions of infinite mixing for the dynamical system (X, \mathcal{A}, μ, T) endowed with the *structure of observables* $(\mathcal{V}, \mathcal{G}, \mathcal{L})$.

The following three definitions will be called **global-local mixing**, as they involve the coupling of a global and a local observable. We say that the system is mixing of type

(GLM1) if, $\forall F \in \mathcal{G}, \forall g \in \mathcal{L}$ with $\mu(g) = 0$, $\lim_{n \rightarrow \infty} \mu((F \circ T^n)g) = 0$;

(GLM2) if, $\forall F \in \mathcal{G}, \forall g \in \mathcal{L}$, $\lim_{n \rightarrow \infty} \mu((F \circ T^n)g) = \bar{\mu}(F)\mu(g)$;

(GLM3) if, $\forall F \in \mathcal{G}$, $\lim_{n \rightarrow \infty} \sup_{g \in \mathcal{L} \setminus 0} \|g\|_1^{-1} |\mu((F \circ T^n)g) - \bar{\mu}(F)\mu(g)| = 0$,

where $\|\cdot\|_1$ is the norm of $L^1(X, \mathcal{A}, \mu; \mathbb{R})$.

Clearly, **(GLM1–3)** are listed in increasing order of strength, with **(GLM2)** being possibly the most natural definition one can give for the time-decorrelation between a global and a local observable (recall that $\bar{\mu}(F \circ T^n) = \bar{\mu}(F)$). **(GLM3)** is a uniform version of it, with important implications (cf. Proposition 2.4), while **(GLM1)** is a much weaker version, as will become apparent in the remainder.

Although this note is mostly concerned with global-local mixing, one can also consider the decorrelation of two global observables, namely **global-global mixing**. For this we need the following terminology:

Definition 2.2 For \mathcal{V} as defined above and $\phi : \mathcal{V} \times \mathbb{N} \rightarrow \mathbb{R}$, we write

$$\lim_{\substack{V \nearrow X \\ n \rightarrow \infty}} \phi(V, n) = \ell$$

to mean

$$\lim_{M \rightarrow \infty} \sup_{\substack{V \in \mathcal{V} \\ \mu(V) \geq M \\ n \geq M}} |\phi(V, n) - \ell| = 0.$$

As n will take the role of time, we refer to this limit as the ‘joint infinite-volume and time limit’.

For $F \in L^\infty$ and $V \in \mathcal{V}$, let us also denote $\mu_V(F) := \mu(V)^{-1} \int_V F d\mu$. We say that the system is mixing of type

(GGM1) if, $\forall F, G \in \mathcal{G}$, $\lim_{n \rightarrow \infty} \bar{\mu}((F \circ T^n)G) = \bar{\mu}(F) \bar{\mu}(G)$;

(GGM2) if, $\forall F, G \in \mathcal{G}$, $\lim_{\substack{V \nearrow X \\ n \rightarrow \infty}} \mu_V((F \circ T^n)G) = \bar{\mu}(F) \bar{\mu}(G)$.

Though **(GGM1)** seems the cleaner of the two versions, it has the serious drawback that, for $n \in \mathbb{N}$, $\bar{\mu}((F \circ T^n)G)$ might not even exist, for there is no provision in our hypotheses to guarantee the ring property for condition (2.2) (namely, $\exists \bar{\mu}(F), \bar{\mu}(G) \Rightarrow \exists \bar{\mu}(FG)$). Nor do we want one, if we are to keep our framework general enough. **(GGM2)** solves this question of wellposedness, and is in some sense stronger than **(GGM1)**:

Proposition 2.3 *If $F, G \in \mathcal{G}$ are such that $\bar{\mu}((F \circ T^n)G)$ exists for all n large enough (depending on F, G), then*

$$\lim_{\substack{V \nearrow X \\ n \rightarrow \infty}} \mu_V((F \circ T^n)G) = \ell \quad \implies \quad \lim_{n \rightarrow \infty} \bar{\mu}((F \circ T^n)G) = \ell. \quad (2.3)$$

*In particular, if the above hypothesis holds $\forall F, G \in \mathcal{G}$, then **(GGM2)** implies **(GGM1)**.*

PROOF. From Definition 2.2, the left limit of (2.3) implies that, $\forall \varepsilon > 0$, $\exists M = M(\varepsilon)$ such that

$$\ell - \varepsilon \leq \mu_V((F \circ T^n)G) \leq \ell + \varepsilon \quad (2.4)$$

for all $V \in \mathcal{V}$ with $\mu(V) \geq M$ and all $n \geq M$. By hypothesis, if M is large enough, the infinite-volume limit of the above middle term exists $\forall n \geq M$ and equals $\bar{\mu}((F \circ T^n)G)$. Upon taking such limit, what is left of (2.4) and its conditions of validity is the very definition of the right limit in (2.3). Q.E.D.

With reasonable hypotheses on the structure of \mathcal{G} and \mathcal{L} , the strongest version of global-local mixing implies the “strongest” version of global-global mixing. The following proposition is a simplified version of a similar result of [L1] (for an intuitive understanding of the hypotheses, see Proposition 3.2 and Remark 3.3 there).

Proposition 2.4 *Suppose there exist a family $(\psi_j)_{j \in \mathbb{N}}$ of real-valued functions of X (this will play the role of a partition of unity) and a family $(\mathbb{J}_V)_{V \in \mathcal{V}}$ of finite subsets of \mathbb{N} such that:*

$$(i) \quad \forall j \in \mathbb{N}, \psi_j \geq 0;$$

(ii) $\forall G \in \mathcal{G}, \forall j \in \mathbb{N}, G\psi_j \in \mathcal{L};$

(iii) in the limit $V \nearrow X$, $\left\| \sum_{j \in \mathbb{J}_V} \psi_j - 1_V \right\|_1 = o(\mu(V)),$

where 1_V is the indicator function of V . Then **(GLM3)** implies **(GGM2)**.

PROOF OF PROPOSITION 2.4. Since the limit in **(GGM2)** is trivial when G is a constant, and since the global observables are bounded functions, it is no loss of generality to prove **(GGM2)** for the case $G \geq 0$ only.

The proof follows upon verification that the functions $g_j := G\psi_j$ verify all the hypotheses of Proposition 3.2 of [L1] (cf. also Remark 3.3). Notice that the identity $G = \sum_j g_j$ (which makes sense insofar as $(\psi_j)_j$ is a partition of unity) is illustrative and not really used in the proof there. Q.E.D.

Since the five definitions presented above deal with the decorrelation of, first, a global and a local observable, and then two global observables, symmetry considerations would induce one to give a definition of **local-local mixing** as well. A reasonable possibility would be to call a dynamical system mixing of type

(LLM) if, $\forall f \in \mathcal{L} \cap \mathcal{G}, g \in \mathcal{L}, \lim_{n \rightarrow \infty} \mu((f \circ T^n)g) = 0.$

In fact, this definition already exists, as it is easy to check that a dynamical system is **(LLM)** if and only if, $\forall A, B \in \mathcal{A}_f, \lim_{n \rightarrow \infty} \mu(T^{-n}A \cap B) = 0$, i.e., if and only if the system is of *zero type* [HK] (cf. also [DS, Ko]). Incidentally, this is the same definition that Krengel and Sucheston call ‘mixing’, for an infinite-measure-preserving dynamical system [KS].

3 Exactness and K-property

Two of the few definitions that are copied verbatim from finite to infinite ergodic theory are those of exactness and K-mixing. Though they are well known, we repeat them here for completeness. We state the versions for measure-preserving maps, but they can be given for nonsingular maps as well (T is nonsingular if $\mu(A) = 0 \Rightarrow \mu(T^{-1}A) = 0$).

Let us denote by \mathcal{N} the *null σ -algebra*, i.e., the σ -algebra that only contains the zero-measure sets and their complements. Also, given two σ -algebras \mathcal{A}, \mathcal{B} , we write $\mathcal{A} = \mathcal{B} \bmod \mu$ if $\forall A \in \mathcal{A}, \exists B \in \mathcal{B}$ with $\mu(A \Delta B) = 0$, and viceversa; equivalently, the μ -completions of \mathcal{A} and \mathcal{B} are the same.

Definition 3.1 *The measure-preserving dynamical system (X, \mathcal{A}, μ, T) is called **exact** if*

$$\bigcap_{n=0}^{\infty} T^{-n}\mathcal{A} = \mathcal{N} \bmod \mu.$$

Since exactness implies that $T^{-1}\mathcal{A} \neq \mathcal{A} \bmod \mu$, a nontrivial exact T cannot be an automorphism of the measure space (X, \mathcal{A}, μ) —although in some sense an invertible map can still be exact, cf. Remark 3.3 below.

The counterpart of exactness for automorphisms is the following:

Definition 3.2 *The invertible measure-preserving dynamical system (X, \mathcal{A}, μ, T) possesses the **K-property** (from A. N. Kolmogorov) if $\exists \mathcal{B} \subset \mathcal{A}$ such that:*

- (i) $\mathcal{B} \subset T\mathcal{B}$;
- (ii) $\bigvee_{n=0}^{\infty} T^n \mathcal{B} = \mathcal{A} \bmod \mu$;
- (iii) $\bigcap_{n=0}^{\infty} T^{-n} \mathcal{B} = \mathcal{N} \bmod \mu$.

In this case, one also says that the dynamical system is *K-mixing*, or that T is a *K-automorphism* of (X, \mathcal{A}, μ) .

Remark 3.3 Comparing Definition 3.1 with condition (iii) of Definition 3.2, one might be tempted to say that, if (X, \mathcal{A}, μ, T) has the K-property, then (X, \mathcal{B}, μ, T) is exact. This is not *technically* correct because, in all nontrivial cases, the inclusion in Definition 3.2(i) is strict, thus T is not a self-map of the measure space (X, \mathcal{B}, μ) . That said, if (X, \mathcal{A}, μ) is a Lebesgue space, (X, \mathcal{B}, μ, T) is still *morally* exact, in the following sense. Let $X_{\mathcal{B}}$ be the measurable partition of X that generates \mathcal{B} , namely the quotient of X w.r.t. the following equivalence relation: $x \sim y$ when, $\forall A \in \mathcal{B}$ with $\mu(A) > 0$, $x \in A \Leftrightarrow y \in A$. By the property of the Lebesgue spaces, \mathcal{B} can be lifted to a σ -algebra for $X_{\mathcal{B}}$, which we keep calling \mathcal{B} . Also, defining $T_{\mathcal{B}}([x]) := [T(x)]$ (where $[\cdot]$ denotes the equivalence classes of \sim), we verify that $T_{\mathcal{B}}$ is well defined as a self-map of $(X_{\mathcal{B}}, \mathcal{B}, \mu)$ (in fact, from Definition 3.2(i), $X_{T\mathcal{B}}$ is a sub-partition of $X_{\mathcal{B}}$) and $T_{\mathcal{B}}^{-1}A = T^{-1}A$, $\forall A \in \mathcal{B}$ (with the understandable abuse of notation whereby A denotes both a subset of $X_{\mathcal{B}}$ and a subset of X). This and Definition 3.2(iii) show that $T_{\mathcal{B}}$ is an exact endomorphism of $(X_{\mathcal{B}}, \mathcal{B}, \mu)$. Of course, in all of the above, \mathcal{B} can be replaced by $\mathcal{B}_m := T^m \mathcal{B}$, for all $m \in \mathbb{Z}$ (because \mathcal{B}_m can be used in Definition 3.2 in lieu of \mathcal{B}).

In finite ergodic theory, both exactness and the K-property imply *mixing of all orders*, namely, $\forall k \in \mathbb{Z}^+$ and $A_1, A_2, \dots, A_k \in \mathcal{A}$,

$$\mu(A_1 \cap T^{-n_2} A_2 \cap \dots \cap T^{-n_k} A_k) \longrightarrow \mu(A_1) \mu(A_2) \cdots \mu(A_k), \quad (3.1)$$

whenever $n_2 \rightarrow \infty$ and $n_{i+1} - n_i \rightarrow \infty$, $\forall i = 2, \dots, k-1$ [Q]. (In (3.1) we have assumed $\mu(X) = 1$.)

One would expect such strong properties to have consequences also in infinite ergodic theory. This is the case, as we describe momentarily. But first we need

some elementary formalism from the functional analysis of dynamical systems. For $F \in L^\infty$ and $g \in L^1$, let us denote

$$\langle F, g \rangle := \mu(Fg). \quad (3.2)$$

Define the *Koopman operator* $U : L^\infty \longrightarrow L^\infty$ as $UF := F \circ T$. Its adjoint for the above coupling is called the *Perron-Frobenius operator*, denoted $P : L^1 \longrightarrow L^1$. Its defining identity is

$$\langle UF, g \rangle = \langle F, Pg \rangle. \quad (3.3)$$

Let us explain in detail how P is defined through (3.3). Take $g \in L^1$ and assume for the moment $g \geq 0$. Take also $F = 1_A$, with $A \in \mathcal{A}$. We see that $\langle UF, g \rangle = \int_{T^{-1}A} g d\mu$. Since T preserves μ and is thus nonsingular w.r.t. it, and since the measure space is σ -finite, the Radon-Nykodim Theorem yields a locally- L^1 , positive, function $Pg : X \longrightarrow \mathbb{R}$ such that $\int_{T^{-1}A} g d\mu = \int_A (Pg) d\mu = \langle F, Pg \rangle$. Using $F = 1_X = 1$, we see that $Pg \in L^1$ with $\|Pg\|_1 = \|g\|_1$. For a general $g \in L^1$, we write $g = g^+ - g^-$, where g^+ and g^- are, respectively, the positive and negative parts of g . Then $Pg := Pg^+ - Pg^-$ is also in L^1 and

$$\|Pg\|_1 \leq \|g\|_1. \quad (3.4)$$

Therefore, through approximations of F via simple functions (on finite-measure sets and in the L^∞ -norm), one can extend (3.3) to all $F \in L^\infty$.

In the process, we have learned that P is a positive operator ($g \geq 0 \Rightarrow Pg \geq 0$) and $\|P\| = 1$, whereas, obviously, U is a positive isometry. Moreover, it is easy to see that $Pg = g$, with $g \geq 0$, if and only if g is an invariant density, i.e., if μ_g defined by $d\mu_g/d\mu = g$ is an invariant measure. (In fact, had we defined (3.2) for $F \in L^1$ and $g \in L^\infty$, (3.3) would have defined a positive operator $P : L^\infty \longrightarrow L^\infty$, with $\|P\| = 1$, and such that $P1 = 1$.)

Most of the reminder of this note will be based on an important theorem by Lin [Li] (see also [A1] for a nice short proof).

Theorem 3.4 *The nonsingular dynamical system (X, \mathcal{A}, μ, T) is exact if and only if, $\forall g \in L^1$ with $\mu(g) = 0$, $\lim_{n \rightarrow \infty} \|P^n g\|_1 = 0$.*

In the rest of the paper we assume to be in one of the following two cases:

- (H1) (X, \mathcal{A}, μ, T) is exact. \mathcal{V} is any exhaustive family that verifies (2.1). $\mathcal{G} = L^\infty$. $\mathcal{L} = L^1$. (Given the assumptions of Section 2, this corresponds to the most general choice of $\mathcal{V}, \mathcal{G}, \mathcal{L}$.)
- (H2) (X, \mathcal{A}, μ, T) is K-mixing (thus T is an automorphism). \mathcal{V} is any exhaustive family that verifies (2.1). \mathcal{G} is the closure, in L^∞ , of $\bigcup_{m \geq 0} L^\infty(\mathcal{B}_m)$, where $\mathcal{B}_m = T^m \mathcal{B}$, as defined in Remark 3.3. Lastly, $\mathcal{L} = L^1$.

Theorem 3.5 *Under either (H1) or (H2),*

- (a) **(GLM1)** *holds true;*
- (b) **(LLM)** *holds true;*
- (c) **(GGM2)** *implies (GLM2);*
- (d) *If, $\forall F \in \mathcal{G}$, $\exists g_F \in \mathcal{L}$, with $\mu(g_F) \neq 0$, such that*

$$\lim_{n \rightarrow \infty} \mu((F \circ T^n)g_F) = \bar{\mu}(F)\mu(g_F),$$

then (GLM2) holds true.

As anticipated in the introduction, **(GLM1)** (which is the most important assertion of the theorem) means that the evolutions of two absolutely continuous initial measures become indistinguishable, as time goes to infinity. We may call this phenomenon *asymptotic coalescence*. This implies that they will return the same measurements of global observables, but not that this measurements will converge (in which case we would have a sort of convergence to equilibrium). In fact, for many interesting systems, it is not hard to construct $F \in L^\infty$ such that $\langle F, P^n g \rangle$ does not converge for all $g \in L^1$.

This is not surprising, for, even in finite ergodic theory, certain proofs of mixing, or decay of correlation, are divided in two parts: asymptotic coalescence and the convergence of *one* initial measure. The difference there is that the latter is usually easy.

The remainder of this section is devoted to the following:

PROOF OF THEOREM 3.5. Let us start by proving assertion (a), namely **(GLM1)**. We use the formalism of functional analysis outlined earlier in the section.

If (H1) is the case, the proof is immediate: for $F \in L^\infty$ and $g \in L^1$, with $\mu(g) = 0$,

$$|\mu((F \circ T^n)g)| = |\langle F, P^n g \rangle| \leq \|F\|_\infty \|P^n g\|_1 \rightarrow 0, \quad (3.5)$$

as $n \rightarrow \infty$, by Theorem 3.4.

In the case (H2), let us observe that, by easy density arguments, all the definitions **(GLM1–3)** hold true if they are verified w.r.t. \mathcal{G}' and \mathcal{L}' which are subspaces of \mathcal{G} and \mathcal{L} , respectively, in the L^∞ - and L^1 -norms. We can take $\mathcal{G}' := \bigcup_{m>0} L^\infty(\mathcal{B}_m)$ (which is dense in \mathcal{G} by definition) and $\mathcal{L}' := \bigcup_{m>0} L^1(\mathcal{B}_m)$, which is dense in $\mathcal{L} = L^1(\mathcal{A})$ by the K-property [A1]. Therefore, it suffices to show **(GLM1)** for a general $m > 0$ and $\forall F \in L^\infty(\mathcal{B}_m)$, $\forall g \in L^1(\mathcal{B}_m)$ with $\mu(g) = 0$.

Using the arguments and the notation of Remark 3.3, we denote by \hat{F} the function induced by F on $X_{\mathcal{B}_m}$ (i.e., $\hat{F}([x]) := F(x)$), and analogously for all the other \mathcal{B}_m -measurable functions. We observe that $F \circ T^n$ is \mathcal{B}_m -measurable and $\widehat{F \circ T^n} = \hat{F} \circ T_{\mathcal{B}_m}^n$. Thus

$$\mu((F \circ T^n)g) = \mu((\hat{F} \circ T_{\mathcal{B}_m}^n)\hat{g}), \quad (3.6)$$

where the r.h.s. is regarded as an integral in $X_{\mathcal{B}_m}$. Since $(X_{\mathcal{B}_m}, \mathcal{B}_m, \mu, T_{\mathcal{B}_m})$ is exact, and $\mu(\hat{g}) = \mu(g) = 0$, we use (3.6) in (3.5) to prove that the l.h.s. of (3.6) vanishes, as $n \rightarrow \infty$.

The following is a corollary of **(GLM1)**.

Lemma 3.6 *Assume either (H1) or (H2), and fix $F \in \mathcal{G}$. If, for some $\ell \in \mathbb{R}$ and $\varepsilon \geq 0$, the limit*

$$\limsup_{n \rightarrow \infty} \left| \frac{\mu((F \circ T^n)g)}{\mu(g)} - \ell \right| \leq \varepsilon$$

holds for some $g \in \mathcal{L}$ (with $\mu(g) \neq 0$), then it holds for all $g \in \mathcal{L}$ (with $\mu(g) \neq 0$).

PROOF OF LEMMA 3.6. Suppose the above limit holds for $g_0 \in \mathcal{L}$. Take any other $g \in \mathcal{L}$, with $\mu(g) \neq 0$. We have:

$$\begin{aligned} & \left| \frac{\mu((F \circ T^n)g)}{\mu(g)} - \ell \right| \\ & \leq \left| \mu \left((F \circ T^n) \left(\frac{g}{\mu(g)} - \frac{g_0}{\mu(g_0)} \right) \right) \right| + \left| \frac{\mu((F \circ T^n)g_0)}{\mu(g_0)} - \ell \right|. \end{aligned} \quad (3.7)$$

By **(GLM1)**, the first term of the above r.h.s. vanishes as $n \rightarrow \infty$, whence the assertion. Q.E.D.

Going back to the proof of Theorem 3.5, we see that Lemma 3.6 immediately implies assertion (d).

As for (b), again we prove it for both cases (H1) and (H2) at the same time. W.l.g., let us assume that $\mathcal{A} \neq \mathcal{N} \bmod \mu$ (otherwise L^1 would be trivial). We claim that

$$\sup_{A \in \mathcal{A}_f} \mu(A) = \infty. \quad (3.8)$$

In fact, since \mathcal{A} is not trivial, the above sup is positive. If it equalled $M \in \mathbb{R}^+$, it would be easy to construct an invariant set B with $0 < \mu(B) \leq M$. But $\mu(X) = \infty$, therefore T would not be ergodic, contradicting both (H1) and (H2).

Now take $f \in L^1 \cap \mathcal{G}$ and $\varepsilon > 0$. By (3.8), $\exists A \in \mathcal{A}_f$ with $\mu(A) \geq \|f\|_1/\varepsilon$. Set $g_\varepsilon = 1_A/\mu(A)$. We have that

$$\left| \frac{\mu((f \circ T^n)g_\varepsilon)}{\mu(g_\varepsilon)} \right| = |\mu((f \circ T^n)g_\varepsilon)| \leq \|f\|_1 \|g_\varepsilon\|_\infty \leq \varepsilon. \quad (3.9)$$

By Lemma 3.6,

$$\limsup_{n \rightarrow \infty} \left| \frac{\mu((f \circ T^n)g)}{\mu(g)} \right| \leq \varepsilon \quad (3.10)$$

holds for all $g \in \mathcal{L}$ with $\mu(g) \neq 0$. Since ε is arbitrary, we get that the above r.h.s. is zero. The case $\mu(g) = 0$ is trivial because the same assertion comes directly from **(GLM1)**. This proves **(LLM)**, namely, assertion (b).

Finally for (c). Take a $G \in \mathcal{G}$ such that $\bar{\mu}(G) > 0$. Since $\mu_V(G) \rightarrow \bar{\mu}(G)$, as $V \nearrow X$, **(GGM2)** implies that there exist a large enough M and a $V \in \mathcal{V}$, with $\mu(V) \geq M$, such that

$$|\mu_V((F \circ T^n)G) - \bar{\mu}(F)\bar{\mu}(G)| \leq \varepsilon \mu_V(G) \quad (3.11)$$

for all $n \geq M$. Setting $g := G1_V$, we can divide (3.11) by $\mu_V(G) = \mu(g)$ and take the lim sup in n :

$$\limsup_{n \rightarrow \infty} \left| \frac{\mu((F \circ T^n)g)}{\mu(g)} - \frac{\bar{\mu}(G)}{\mu_V(G)} \bar{\mu}(F) \right| \leq \varepsilon. \quad (3.12)$$

By Lemma 3.6, the above holds $\forall g \in \mathcal{L}$, with $\mu(g) \neq 0$. Since ε can be taken arbitrarily close to 0 and $\bar{\mu}(G)/\mu_V(G)$ arbitrarily close to 1, we have that, for all $F \in \mathcal{G}$ and $g \in \mathcal{L}$, with $\mu(g) \neq 0$,

$$\lim_{n \rightarrow \infty} \mu((F \circ T^n)g) = \bar{\mu}(F)\mu(g). \quad (3.13)$$

The corresponding statement for $\mu(g) = 0$ comes from **(GLM1)**.

Q.E.D.

4 The equilibrium observables

The “pure” ergodic theorist might raise an eyebrow at the constructions of Section 2, especially at the ideas of the exhaustive family (which demands that one singles out some sets as more important than the others) and of the infinite-volume average (which is not a measure, or even guaranteed to always exist).

Though these issues (and more) have been addressed in [L1], one might still want to see if some of the concepts presented here can be viewed from the vantage point of traditional infinite ergodic theory. For what follows I am indebted to R. Zweimüller.

As we discussed in the introduction, the definition **(GLM2)** makes sense as a kind of convergence to equilibrium for a large class of initial distributions (see also the observation on **(GLM1)** after the statement of Theorem 3.5). Without worrying too much about predetermining good test functions for this convergence (namely, the global observables), and the value of any such limit (namely, the infinite-volume average), one might simply consider the space $\mathcal{E} = \mathcal{E}(X, \mathcal{A}, \mu, T)$ of *all* the good test functions, in this sense:

$$\mathcal{E} := \left\{ F \in L^\infty \mid \exists \rho(F) \in \mathbb{R} \text{ s.t. } \lim_{n \rightarrow \infty} \mu((F \circ T^n)g) = \rho(F)\mu(g), \forall g \in L^1 \right\}. \quad (4.1)$$

(Occasionally, one might want to restrict the space of the initial distributions to some subspace of L^1 .) Clearly, \mathcal{E} is a vector space which contains at least the constant functions.

$\rho(F)$ represents a sort of *value at equilibrium* of F and, in this context, it need not have anything to do with $\bar{\mu}(F)$ (which might or might not exist), \mathcal{V} , or the

choice of \mathcal{G} and \mathcal{L} . Thus, the elements of the vector space \mathcal{E} may be called the **equilibrium observables** and $\rho : \mathcal{E} \rightarrow \mathbb{R}$ the **equilibrium functional**.

If we are in either case (H1) or (H2), Theorem 3.5(d) shows that, for a given $F \in \mathcal{G}$, one only need find *one* local observable that verifies the limit in (4.1). Also, by Theorem 3.5(b), any $f \in \mathcal{G} \cap L^1$ belongs to \mathcal{E} , with $\rho(f) = 0$. Therefore, in these cases, it makes sense to introduce $[\mathcal{E}] := \mathcal{E}/(\mathcal{G} \cap L^1)$, and ρ is well-defined there. When talking about $[\mathcal{E}]$, we write $F \in [\mathcal{E}]$ to mean $F \in \mathcal{E}$, and $F = G$ to mean $[F] = [G]$.

Determining $[\mathcal{E}]$ for a given, say, exact dynamical system appears to be as complicated as proving **(GLM2)** for a truly large class of global observables, though occasionally some information can be obtained quickly. We conclude this note by giving some examples thereof.

Boole Transformation. This is the transformation $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) := x - 1/x$. This map preserves the Lebesgue measure on \mathbb{R} , as it is easy to verify, and is exact [A1]. We can use the fact that T is odd to construct a nonconstant equilibrium observable. Set $F(x) := \text{sign}(x)$, and $g := 1_{[-1,1]}$. Clearly, for all $n \in \mathbb{N}$, $F \circ T^n$ is odd and $\mu((F \circ T^n)g) = 0$, so $F \in [\mathcal{E}]$, with $F \neq \text{constant}$, and $\rho(F) = 0$.

Evidently, the same reasoning can be applied to any exact map with an odd symmetry.

Translation-invariant expanding maps of \mathbb{R} . Take a C^2 bijection $\Phi : [0, 1] \rightarrow [k_1, k_2]$, with $k_1, k_2 \in \mathbb{Z}$, and $\Phi' > 1$, where Φ' denotes the derivative of Φ . (Notice that these conditions imply $\Phi(0) = k_1$, $\Phi(1) = k_2$, and $k := k_2 - k_1 \geq 2$.) Define $T : \mathbb{R} \rightarrow \mathbb{R}$ via

$$T|_{[j, j+1)}(x) := \Phi(x - j) + j, \quad (4.2)$$

for all $j \in \mathbb{Z}$. By construction $T(x + 1) = T(x) + 1$, $\forall x \in \mathbb{R}$, and so T is a k -to-1 translation-invariant map, in the sense that it commutes with the natural action of \mathbb{Z} in \mathbb{R} .

Suppose that T preserves the Lebesgue measure, which we denote $m_{\mathbb{R}}$. (One can easily construct a large class of maps of this kind.) It can be proved that any such T is exact [L3]. Now, define $I := [0, 1)$ and $T_I : I \rightarrow I$ as $T_I(x) := T(x) \bmod 1$. Clearly, $(I, \mathcal{B}_I, T_I, m_I)$, where \mathcal{B}_I and m_I are, respectively, the Borel σ -algebra and the Lebesgue measure on I , is a probability-preserving dynamical system. It is easy to see that it is exact, and thus mixing.

Now consider a \mathbb{Z} -periodic, bounded, $F : \mathbb{R} \rightarrow \mathbb{R}$. Evidently, $\forall x \in I$, $\forall n \in \mathbb{N}$, $F \circ T^n(x) = F \circ T_I^n(x)$. Hence, by the mixing of the quotient dynamical system, for any square-integrable g supported in I ,

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{\mathbb{R}}((F \circ T^n)g) &= \lim_{n \rightarrow \infty} m_I((F \circ T_I^n)g) \\ &= m_I(F) m_I(g) \\ &= m_I(F) m_{\mathbb{R}}(g). \end{aligned} \quad (4.3)$$

By the exactness of T , the above holds for all $g \in L^1(\mathbb{R})$. Hence $F \in [\mathcal{E}]$, with $\rho(F) = m_I(F) = \overline{m}_{\mathbb{R}}(F)$.

An analogous procedure (using $I_j := [0, j)$ instead of I) can be employed to prove that any $(j\mathbb{Z})$ -periodic, bounded F belongs in $[\mathcal{E}]$, with $\rho(F) = \overline{m}_{\mathbb{R}}(F)$. In [L3] we extend this result to observables that are quasi-periodic w.r.t. any $j\mathbb{Z}$, and more.

Random walks. A special case of the above situation occurs when Φ is linear. The result is a piecewise linear Markov map that represents a random walk in \mathbb{Z} , in the following sense. Denote by $\lfloor x \rfloor$ the maximum integer not exceeding $x \in \mathbb{R}$. If an initial condition $x \in I$ is randomly chosen with law m_I , then the stochastic process $(\lfloor T^n(x) \rfloor)_{n \in \mathbb{N}}$ is precisely the random walk starting in $0 \in \mathbb{Z}$, with uniform transition probabilities for jumps of $k_1, k_1 + 1, \dots, k_2 - 1$ units [L2].

A reelaboration of a result of [L1] shows that $[\mathcal{E}]$ contains all L^∞ functions such that the limit

$$\rho(F) := \lim_{M \rightarrow \infty} \int_{a-M}^{a+M} F(x) dx \quad (4.4)$$

exists independently of and uniformly in $a \in \mathbb{R}$. In fact, it is proved in [L1, Thm. 4.6(b)] (see also [L2, Thm. 9]) that, if $F \in L^\infty(\mathcal{A}_0)$, where \mathcal{A}_0 is the σ -algebra generated by the partition $\{[j, j+1)\}_j$, and $g \in L^1$, then $m_{\mathbb{R}}((F \circ T^n)g) \rightarrow \overline{m}_{\mathbb{R}}(F)m_{\mathbb{R}}(g)$, as $n \rightarrow \infty$, with

$$\overline{m}_{\mathbb{R}}(F) = \lim_{j \rightarrow \infty} \int_{q-j}^{q+j} F(x) dx, \quad (4.5)$$

$j \in \mathbb{Z}$, and the limit is attained uniformly in $q \in \mathbb{Z}$. Obviously, in view of (4.4), $\rho(F) = \overline{m}_{\mathbb{R}}(F)$.

Now, for a general F , one can take $g = 1_{[0,1)} \in L^1(\mathcal{A}_0)$. It is easy to check that $P^n g$ is \mathcal{A}_0 -measurable too, thus

$$\begin{aligned} \lim_{n \rightarrow \infty} m_{\mathbb{R}}((F \circ T^n)g) &= \lim_{n \rightarrow \infty} \langle \mathbb{E}(F|\mathcal{A}_0), P^n g \rangle \\ &= \overline{m}_{\mathbb{R}}(\mathbb{E}(F|\mathcal{A}_0)) m_{\mathbb{R}}(g) \\ &= \rho(F) m_{\mathbb{R}}(g), \end{aligned} \quad (4.6)$$

which proves our claim.

If the random walk has a drift, say a positive drift, then a.e. orbit will converge to $+\infty$. Therefore, any bounded function G that asymptotically shadows any of the above observables—meaning $\lim_{x \rightarrow +\infty} (G(x) - F(x)) = 0$, for some F verifying (4.4)—will also belong to $[\mathcal{E}]$, with $\rho(G) = \rho(F)$.

References

- [A1] J. AARONSON, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.

- [A2] J. AARONSON, *Rational weak mixing in infinite measure spaces*, to appear in Ergodic Theory Dynam. Systems (2013).
- [AMPS] A. ARBIETO, R. MARKARIAN, M. J. PACIFICO AND R. SOARES, *Scaling rate for semi-dispersing billiards with non-compact cusps*, Ergodic Theory Dynam. Systems **32** (2012), no. 6, 1836–1861.
- [DR] A. I. DANILENKO AND V. V. RYZHIKOV, *Mixing constructions with infinite invariant measure and spectral multiplicities*, Ergodic Theory Dynam. Systems **31** (2011), no. 3, 853–873.
- [DS] A. I. DANILENKO AND C. E. SILVA, *Ergodic Theory: nonsingular transformation*, in: R. E. Meyers (ed.), Encyclopedia of Complexity and Systems Science, pp. 3055–3083, Springer, 2009.
- [HK] A. B. HAJIAN AND S. KAKUTANI, *Weakly wandering sets and invariant measures*, Trans. Amer. Math. Soc. **110** (1964), 136–151.
- [I1] S. ISOLA, *Renewal sequences and intermittency*, J. Statist. Phys. **97** (1999), no. 1-2, 263–280.
- [I2] S. ISOLA, *On systems with finite ergodic degree*, Far East J. Dyn. Syst. **5** (2003), no. 1, 1–62.
- [I3] S. ISOLA, *From infinite ergodic theory to number theory (and possibly back)*, Chaos Solitons Fractals **44** (2011), no. 7, 467–479.
- [Ko] Z. KOSLOFF, *The zero-type property and mixing of Bernoulli shifts*, to appear in Ergodic Theory Dynam. Systems (2013).
- [KS] U. KRENGEL AND L. SUCHESTON, *Mixing in infinite measure spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **13** (1969), 150–164.
- [LP] M. LEMAŃCZYK AND F. PARREAU, *Lifting mixing properties by Rokhlin cocycles*, Ergodic Theory Dynam. Systems **32** (2012), no. 2, 763–784.
- [L1] M. LENCI, *On infinite-volume mixing*, Comm. Math. Phys. **298** (2010), no. 2, 485–514.
- [L2] M. LENCI, *Infinite-volume mixing for dynamical systems preserving an infinite measure*, Procedia IUTAM **5** (2012), 204–219.
- [L3] M. LENCI, *Infinite mixing for piecewise expanding maps of the real line*, in preparation.
- [Li] M. LIN, *Mixing for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **19** (1971), 231–242.
- [MT] I. MELBOURNE AND D. TERHESIU, *Operator renewal theory and mixing rates for dynamical systems with infinite measure*, Invent. Math. **189** (2012), no. 1, 61–110.
- [Q] A. QUAS, *Ergodicity and mixing properties*, in: R. E. Meyers (ed.), Encyclopedia of Complexity and Systems Science, pp. 2918–2933, Springer, 2009.
- [T1] D. TERHESIU, *Improved mixing rates for infinite measure preserving systems*, preprint (2012).
- [T2] D. TERHESIU, *Mixing rates for intermittent maps of high exponent*, preprint (2012).
- [Z] R. ZWEIMÜLLER, *Mixing limit theorems for ergodic transformations*, J. Theoret. Probab. **20** (2007), no. 4, 1059–1071.